

Volume fluctuations and higher order cumulants of the net baryon number

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We consider the effect of volume fluctuations on cumulants of the net baryon number. Based on a general formalism, we derive universal expressions for the net baryon number cumulants in the presence of volume fluctuations with an arbitrary probability distribution. The relevance of these fluctuations for the baryon-number cumulants and in particular for the ratios of cumulants is assessed in the Polyakov loop extended quark-meson model within the functional renormalization group. We show that the baryon number cumulants are generally enhanced by volume fluctuations and that the critical behavior of higher order cumulants may be modified significantly.

I. INTRODUCTION

One of the goals of the experiments with ultrarelativistic heavy ion collisions at SPS, RHIC and LHC energies is to probe the phase structure of strongly interacting matter and, in particular, to identify the deconfinement and chiral restoration transitions. In this context, the fluctuations of conserved charges may serve as a pertinent probe.

Fluctuations of the net baryon number and electric charge may provide an experimental signature for the hypothetical chiral critical endpoint [1, 2]. Moreover, as recently noted [3–11], such fluctuations are also of interest at small baryon densities, since they reflect the critical dynamics of the underlying $O(4)$ transition, expected in QCD in the limit of massless light quarks [12, 13]. Indeed, it was demonstrated that higher order cumulants change sign in the crossover region of the QCD phase diagram [2, 8, 9]. Thus, the observation of a strong suppression the higher order cumulants may be used to identify the chiral crossover transition in experiment.

The first measurements of fluctuations of the net baryon number, more precisely of the net proton number¹, in heavy ion collisions at RHIC were obtained by the STAR Collaboration [15]. The analysis of cumulants of the fluctuations and of the probability distributions confirmed, that the hadron resonance gas (HRG) model, which yields a quantitative description of particle yields in heavy ion collisions [16], provides a useful reference for the non-critical background contribution to the charge fluctuations [4]. Thus, critical fluctuations related to the dynamics of the chiral transition should be reflected in deviations of the measured net charge fluctuations from the HRG baseline. In this context, higher order cumulants are of particular interest [8, 10].

A detailed analysis of experimental data on moments of net proton number fluctuations and their probability

distributions indeed exhibit deviations from the HRG. To verify the origin of these deviations, one must identify and assess effects, unrelated to the critical dynamics, which can influence the charge fluctuations. For instance, it was recently argued that constraints, owing to the conservation of the total baryon number in nucleus-nucleus collisions, modify the non-critical background contributions to higher order cumulants of the net proton number fluctuations [17].

In this paper we study volume fluctuations as a further possible source of non-critical fluctuations, not accounted for in the HRG model results. We first present a transparent derivation of the cumulants of net baryon number, including the effect of volume fluctuations. The resulting cumulants are expressed in terms of cumulants of the net baryon number distribution at fixed volume and cumulants of the probability distribution for volume fluctuations. We also provide a more formal derivation, making use of the cumulant generating functions. We stress that the final expressions are general, independent of the probability distributions for net baryon number and volume. The only assumption made is that the two sources of fluctuations are independent.

Furthermore, we assess the effect of volume fluctuations on the kurtosis and on ratios of higher order cumulants of the net baryon number in the vicinity of the chiral transition within the Polyakov loop extended quark-meson model. Here we employ the functional renormalization group to properly account for critical fluctuations.

The paper is organized as follows: In the next section we obtain the corrections due to volume fluctuations to the first four moments of the net baryon number fluctuations. In Section III we derive a general expression for the corrected cumulants, valid to any order, obtained using the cumulant generating functions. In Section IV we illustrate the role of volume fluctuations with a numerical study and finally in Section V we state our conclusions.

II. HEURISTIC APPROACH

Consider a fixed volume V , where the net baryon number B fluctuates with the probability distribution

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¹ As shown recently in Ref. [14], the net nucleon number cumulants can, to a good approximation, be deduced from the measured net proton cumulants.

$P(B, V)$. The n -th order moments of the net baryon number are then defined by

$$\langle B^n \rangle_V = \sum_{B=-\infty}^{\infty} B^n P(B, V). \quad (1)$$

It is convenient to introduce reduced cumulants, corresponding to the net baryon number fluctuations per unit volume. The first four reduced cumulants are

$$\begin{aligned} \kappa_1(T, \mu) &= \frac{1}{V} \langle B \rangle_V, \\ \kappa_2(T, \mu) &= \frac{1}{V} \langle (\delta B)^2 \rangle_V, \\ \kappa_3(T, \mu) &= \frac{1}{V} \langle (\delta B)^3 \rangle_V, \\ \kappa_4(T, \mu) &= \frac{1}{V} [\langle (\delta B)^4 \rangle_V - 3\langle (\delta B)^2 \rangle_V^2], \end{aligned} \quad (2)$$

where $\delta B = B - \bar{B}$. The cumulants κ_i are, to leading order, independent of the volume V . In the following we neglect subleading surface effects, which could lead to a residual volume dependence of the cumulants.

The volume dependence of the moments follows from (2) and reads

$$\begin{aligned} \langle B \rangle_V &= \kappa_1 V, \\ \langle B^2 \rangle_V &= \kappa_2 V + \kappa_1^2 V^2, \\ \langle B^3 \rangle_V &= \kappa_3 V + 3\kappa_2 \kappa_1 V^2 + \kappa_1^3 V^3, \\ \langle B^4 \rangle_V &= \kappa_4 V + (4\kappa_3 \kappa_1 + 3\kappa_2^2) V^2 + 6\kappa_2 \kappa_1^2 V^3 + \kappa_1^4 V^4. \end{aligned} \quad (3)$$

The coefficients in Eq. (3) are those of the Bell polynomials.

As an illustrative example, we consider the hadron resonance gas. In this model, the net baryon number fluctuations are given by the Skellam distribution [5, 6] and the corresponding cumulants are particularly simple²

$$\kappa_{2n+1}^{(HRG)} = \frac{1}{V} (\bar{B}_1 - \bar{B}_{-1}), \quad \kappa_{2n}^{(HRG)} = \frac{1}{V} (\bar{B}_1 + \bar{B}_{-1}), \quad (4)$$

where $\bar{B}_1 = \langle B_1 \rangle$ is the mean number of baryons and $\bar{B}_{-1} = \langle B_{-1} \rangle$ that of anti-baryons in V . The corresponding moments are obtained by inserting the cumulants (4) in (3).

We now allow for fluctuations of the volume. To this end, we introduce the volume probability distribution $\mathcal{P}(V)$, the corresponding moments

$$\langle V^n \rangle = \int V^n \mathcal{P}(V) dV, \quad (5)$$

and the reduced cumulants of the volume fluctuations, v_n . The latter are defined as in Eq. (2) with the replacements $V \rightarrow \langle V \rangle$ and $B \rightarrow V$. Thus, e.g. $v_1 = 1$ and $v_2 = (\langle V^2 \rangle - \langle V \rangle^2) / \langle V \rangle$.

In the presence of volume fluctuations the moments of the net baryon number are given by

$$\begin{aligned} \langle B^n \rangle &= \int dV \mathcal{P}(V) \sum_{B=-\infty}^{\infty} B^n P(B, V) \\ &= \int dV \mathcal{P}(V) \langle B^n \rangle_V. \end{aligned} \quad (6)$$

It is now straightforward to compute the reduced cumulants, including the effect of volume fluctuations. Using Eqs. (2), (3) and (6), we find the general relations

$$\begin{aligned} c_1 &= \kappa_1 \\ c_2 &= \kappa_2 + \kappa_1^2 v_2, \\ c_3 &= \kappa_3 + 3\kappa_2 \kappa_1 v_2 + \kappa_1^3 v_3, \\ c_4 &= \kappa_4 + (4\kappa_3 \kappa_1 + 3\kappa_2^2) v_2 + 6\kappa_2 \kappa_1^2 v_3 + \kappa_1^4 v_4, \end{aligned} \quad (7)$$

which are valid for arbitrary probability distributions, provided the fluctuations in baryon number and volume are independent. We note that the form of (7) is determined by the volume dependence of the moments (3). Hence, the coefficients in (7) are also given by the Bell polynomials.

III. GENERAL DERIVATION

In the previous section, we explored the effect of volume fluctuations on the fluctuations of the net baryon number for the first few cumulants, where explicit calculations are tractable. In the following we derive a general expression for the cumulants, under the assumption that the fluctuations of baryon number and volume are independent.

A. Formalism

In general, the probability distributions introduced in section II are characterized by the corresponding cumulant generating functions³,

$$\chi^B(t) = \ln \sum_{B=-\infty}^{\infty} P(B) \exp(Bt), \quad (8)$$

$$\chi^V(s) = \ln \int_0^{\infty} dV \mathcal{P}(V) \exp(Vs). \quad (9)$$

The cumulants are obtained by expanding χ^B and χ^V in series about the origin. The additivity of cumulants and

² The normalization of the generalized susceptibilities given in [6] differs from the cumulants used here by a factor T^3 .

³ We assume that the integrals in (8) and (9) converge for t and s in an interval around the origin, so that the cumulant generating functions exist [18]. Moreover, we assume that the relevant cumulants exist.

thermodynamic principles imply that

$$\chi^B(t) = V \cdot \zeta^B(t), \quad (10)$$

where ζ^B is a volume-independent function. In fact, ζ^B is the generating function for the reduced cumulants, defined in Eqs. (2):

$$\kappa_n = \frac{d^n}{dt^n} \zeta^B(t) \Big|_{t=0}. \quad (11)$$

Similarly, we find for the reduced cumulants of volume fluctuations

$$v_n = \frac{1}{\langle V \rangle} \frac{d^n}{dt^n} \chi^V(t) \Big|_{t=0}. \quad (12)$$

Our aim is to compute cumulants of the net baryon number *including* the effects of volume fluctuations. These cumulants are obtained from the cumulant generating function

$$\phi^B(t) = \ln \int dV \mathcal{P}(V) \sum_B P(B, V) e^{Bt}. \quad (13)$$

Using Eq. (10) we find

$$\sum_B P(B, V) e^{Bt} = e^{V\zeta^B(t)}, \quad (14)$$

and consequently

$$\phi^B(t) = \ln \int dV \mathcal{P}(V) e^{V\zeta^B(t)}. \quad (15)$$

A comparison with the definition of the cumulant generating function (9), yields

$$\phi^B(t) = \chi^V [\zeta^B(t)]. \quad (16)$$

This is the general form of the cumulant generating function for fluctuations of the net baryon number, including the effect of volume fluctuations. The corresponding reduced cumulants are given by a Taylor expansion of $\phi^B(t)$ about $t = 0$,

$$c_n = \frac{1}{\langle V \rangle} \frac{d^n}{dt^n} \phi^B(t) \Big|_{t=0}. \quad (17)$$

We note that since $\zeta^B(t=0) = 0$, no further normalization is needed in the calculation of the cumulants.

Using Faà di Bruno's formula [19], we obtain a closed form expression for the cumulants,

$$c_n = \sum_{i=1}^n v_n B_{n,i}(\kappa_1, \kappa_2, \dots, \kappa_{n-i+1}), \quad (18)$$

where $B_{n,i}$ are Bell polynomials. This equation confirms and extends our previous results for the first four cumulants, given in Eq. (7). Thus, for an arbitrary probability distribution for the fluctuations of net baryon number as well as for the fluctuations of the volume, Eq. (18) yields cumulants that can be confronted with experiment. Conversely, given a model for the volume fluctuations, Eq. (18) can be used to extract cumulants of the net baryon number in a fixed volume.

B. Gaussian volume fluctuations

If the probability distribution for volume fluctuations is approximately Gaussian, i.e. $v_n \ll 1$ for $n > 2$, the higher order cumulants are simplified considerably

$$\begin{aligned} c_3^G &= \kappa_3 + 3\kappa_1\kappa_2v_2, \\ c_4^G &= \kappa_4 + (3\kappa_2^2 + 4\kappa_1\kappa_3)v_2, \\ c_6^G &= \kappa_6 + (10\kappa_3^2 + 15\kappa_2\kappa_4 + 6\kappa_1\kappa_5)v_2, \\ c_8^G &= \kappa_8 \\ &\quad + (35\kappa_4^2 + 56\kappa_3\kappa_5 + 28\kappa_2\kappa_6 + 8\kappa_1\kappa_7)v_2. \end{aligned} \quad (19)$$

In the section IV we use this form to explore the effect of volume fluctuations on the cumulants of net baryon number.

IV. NUMERICAL RESULTS IN THE PQM MODEL

We illustrate the influence of volume fluctuations on net baryon number fluctuations, within a model calculation. Of particular interest is the modification of higher order moments near the chiral crossover transition. We adopt the Polyakov loop-extended Quark Meson Model (PQM) and compute the cumulants in a non-perturbative scheme, the functional renormalisation group. Details on the calculations and on the derivation of the net baryon number fluctuations can be found in Ref. [9]. In this exploratory calculation, we consider only Gaussian fluctuations of the volume for a system at zero baryon chemical potential. In this case, all odd cumulants of the net baryon number vanish, i.e. $c_{2n+1} = \kappa_{2n+1} = 0$.

In Ref. [8] it was shown that near the chiral crossover transition, higher cumulants of the net baryon number ($n > 4$) differ considerably from the predictions of the HRG model. In particular, it was suggested that negative values of κ_6 and κ_8 could be used to map out the chiral phase boundary. This potential signal for the QCD phase transition may be affected by volume fluctuations. Indeed, the second term in Eq. (19) yields a positive contribution to c_6 . The strength of this contribution is directly proportional to the second cumulant v_2 .

In general, volume fluctuations are difficult to assess. In heavy ion collisions they depend on the centrality of the collision, on the definition used to fix the number of participants and on the kinematical window, where the fluctuations are measured. Thus, v_2 is specific to a given experimental setup. We choose $v_2 = 0.6$, employed in Ref. [20].

It is useful to consider ratios of cumulants,

$$R_{n,m} = \frac{c_n}{c_m}, \quad (20)$$

where the dependence on the (mean) volume of the subsystem cancels. In Figs. 1 and 2 we show the temperature-dependence of the $R_{4,2}$ and $R_{6,2}$ obtained

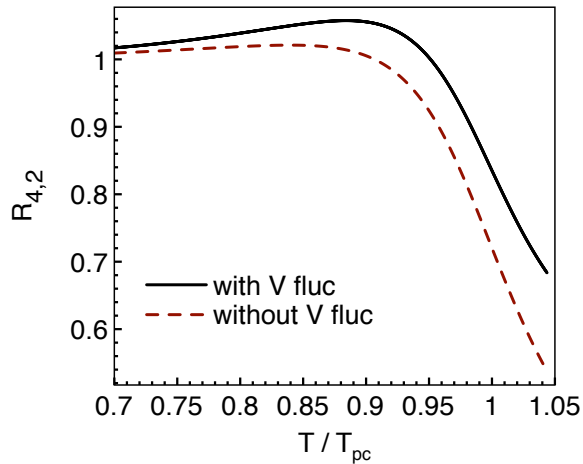


FIG. 1: The ratio $R_{4,2}$ as a function of temperature computed in the PQM model at vanishing chemical potential with and without volume fluctuations. The probability distribution for volume fluctuations is assumed to be Gaussian with variance $v_2 = 0.6$, see the text for details.

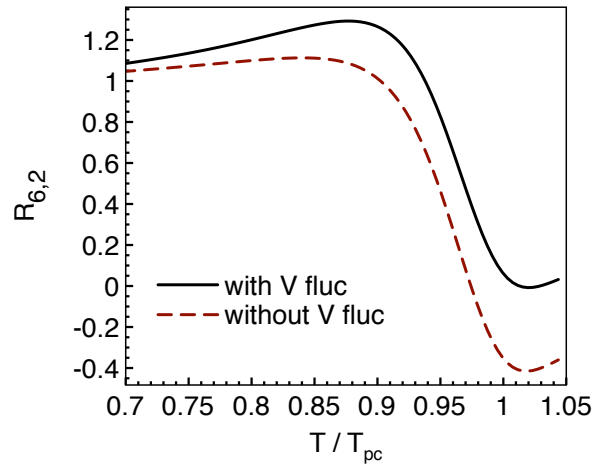


FIG. 2: Same as Fig. 1, but for $R_{6,2}$.

in the PQM model for finite and vanishing v_2 . Moreover, in Fig. 3, the dependence of the ratios $R_{n,m}$ on v_2 at the crossover transition temperature is illustrated.

These results show that volume fluctuations tend to suppress the signature of the chiral transition in the cumulants of net baryon number. Here, the ratio $R_{6,2}$ seems to be particularly sensitive. Consequently, the usefulness of fluctuations of conserved charges as a probe of criticality in heavy ion collisions, depends crucially on the possibility to control volume fluctuations.

V. CONCLUSIONS

We have studied the influence of volume fluctuations on the properties of cumulants of net charge distributions in heavy ion collisions. In particular, we have computed the contribution of volume fluctuations to cumulants of the net baryon number.

In a heuristic approach we showed explicitly how the corrections due to volume fluctuations arise. The expressions, which hold for arbitrary probability distributions, were confirmed and extended in a general formalism, where we employed the cumulant generating functions to obtain a closed form for the cumulants, including volume fluctuations.

We assessed the contribution of volume fluctuations for the kurtosis $R_{4,2}$ as well as for ratios involving higher order cumulants, *viz.* $R_{6,2}$ and $R_{8,2}$, in the Polyakov loop extended quark-meson model. A non-perturbative treatment of fluctuations was obtained by employing the functional renormalisation group. We focused on the structure of the ratios of cumulants in the vicinity to the chiral crossover transition, assuming that the probability distribution of volume fluctuations is approximately Gaussian.

Finally, we showed that phenomenologically relevant ratios of cumulants of the net baryon number are enhanced by volume fluctuations. Consequently, for a sufficiently large v_2 , the negative structure of these ratios, expected in heavy ion collisions owing to the critical chiral dynamics, may be modified significantly by volume fluctuations. It follows that fluctuations of conserved charges in heavy ion collisions can provide a robust probe of the chiral phase boundary, if good control of volume fluctuations can be achieved.

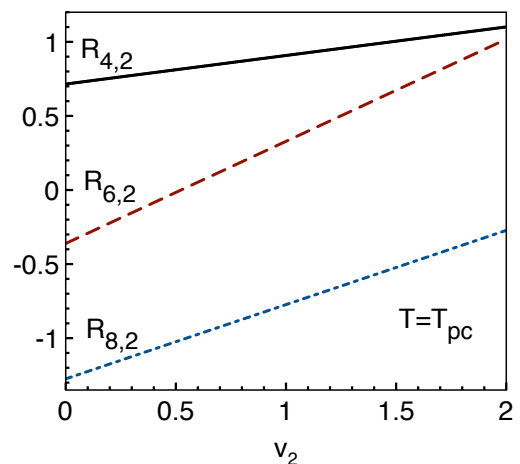


FIG. 3: The ratios $R_{n,m}$ at the chiral crossover temperature T_{pc} , obtained in the PQM model, as functions of the second cumulant of the volume fluctuations, v_2 .

Acknowledgments

We thank P. Braun-Munzinger for stimulating discussions. The research of V.S. was supported under Contract No. DE-AC02-98CH10886 with the U. S. Depart-

ment of Energy. B.F. and K.R. are supported in part by the ExtreMe Matter Institute EMMI. K.R. acknowledges partial support by the Polish Ministry of National Education (MEN).

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